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Raney and Catalan

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Abstract

Raney's lemma is often used in a counting argument to prove the formula for (generalized) Catalan numbers. It ensures the existence of "good" cyclic shifts of certain sequences, i.e. cyclic shifts for which all partial sums are positive.

We introduce a simple algorithm that finds these cyclic shifts and also those with a slightly weaker property. Moreover it provides simple proofs of lemma's of Raney type.

A similar clustering procedure is also used in a simple proof of a theorem on probabilities of which many well-known results (e.g. on lattice paths and on generalized Catalan numbers) can be derived as corollaries. The theorem generalizes generalized Catalan numbers. In the end it turns out to be equivalent to a formula of Raney.

Keywords: Catalan numbers; Raney's lemma

1. Splitting points and Raney's lemma

1.1. Finding "good" cyclic shifts

By cyclic shifts a finite sequence $a_1, a_2, \ldots, a_n$ (always of reals) can be transformed into $a_i, a_{i+1}, \ldots, a_n, a_1, a_2, \ldots, a_{i-1}$. We are concerned with the partial sums of these sequences. Let S_i^j denote $\sum_{t=i}^{j} a_t$ if $i \le j \le n$ and $\sum_{t=i}^{n} a_t + \sum_{t=1}^{j} a_t$ if $1 \le j \le i-1$. We call i a splitting point (a weak splitting point) of $a_1, a_2, \ldots, a_n$ if $S_i^j > 0$ (≥ 0) for all $j \ne i-1$ (if $i = 1$ read $j \ne n$). Of course one visualizes this "point" between a_{i-1} and a_i. Excluding here the "full" partial sum $S = S_i^{i-1}$ (as well as the empty one) has no consequences for sequences with a positive sum, but in this way also other ones may have a (weak) splitting point. That now every sequence of length 1 has 1 as a splitting point is harmless.

If $S < 0$ there is at most one weak splitting point: if $S_i^j \ge 0$ for all j with $j \ne i-1$ then $S_k^{i-1} = S - S_i^{k-1} < 0$ for all $k \ne i$. If $S = 0$ there may be more than one weak splitting point, unless there is a splitting point.

Suppose $n > 1$ and $a_j < 0$. Let $b_1, b_2, \ldots, b_n$ be the sequence with $b_{j-1} = a_{j-1} + a_j$, $b_j = 0$, $b_t = a_t$ otherwise (read $j-1$ as n if $j = 1$). We shall say that it is derived from $a_1, a_2, \ldots, a_n$ by a *push* at j. The first lemma is trivial (as to item d); note that a push at j implies that j cannot be a (weak) splitting point: "one cannot push past a weak splitting point").

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Lemma 1. *Let $b_1, b_2, \ldots, b_n$ be derived from $a_1, a_2, \ldots, a_n$ by a push. Then*

(a) $\sum_{t=1}^{n} a_t = \sum_{t=1}^{n} b_t$.
(b) *The b_t are integers if the a_t are.*
(c) *If $m > 0$ and $a_t \le m$ for all t, then $b_t \le m$ for all t.*
(d) *If r and s are weak splitting points of $a_1, a_2, \ldots, a_n$, then $S_r^{s-1} a_t = S_r^{s-1} b_t$.*

Lemma 2. *If $b_1, b_2, \ldots, b_n$ is derived from $a_1, a_2, \ldots, a_n$ by a push, and if $\sum_{i=1}^{n} a_i > 0$, then the two sequences have the same splitting points.*

Proof. Suppose the push was at j. Then $a_j < 0$ and $b_j = 0$, so j is a splitting point of neither sequence. For $i \ne j$ we have: $S_i^k a_t = S_i^k b_t$ if $k \ne j-1$, $S_i^{j-1} a_t = S_i^{j-1} b_t - a_j > S_i^{j-1} b_t$ and $S_i^{j-1} b_t = S_i^{j} a_t$. It follows that i is a splitting point of both sequences or of neither of them. ($\sum_{i=1}^{n} a_i > 0$ is needed for the case $i = j+1$, cf. the sequence 1, 1, −2.) □

A sequence $c_1, c_2, \ldots, c_n$ will be called *reduced* if one of the following holds:

1. $c_1 < 0$, $c_i = 0$ for $i > 1$,
2. $c_i = 0$ for all i,
3. $c_i \ge 0$ for all i, $c_i > 0$ for some i.

Lemma 3. *Every sequence can be transformed into a reduced sequence by a series of pushes. That reduced sequence is unique.*

Proof. A sequence $a_1, a_2, \ldots, a_n$ can be transformed into a reduced sequence $c_l, c_2, \ldots, c_n$ for instance by first pushing successively at $n, n-1, \ldots, 2$ if a negative number occurs there (the first "sweep"). If then the first term is negative and there still are positive terms, we push at 1 (the "switch") and again, while still possible, at $n, n-1, \ldots, 2$ (the second "sweep"). Now by Lemma 1(a) we arrive at a reduced sequence of type (1), (2), (3), respectively, if the sum of the given sequence is negative, zero, positive, respectively. In the first two cases the uniqueness is trivial. In the third case the s for which $c_s > 0$ are the splitting points, also, by Lemma 2, of the starting sequence. By Lemma 1(d) $c_s = S_s^{r-1} a_t$ if r is the splitting point following s in cyclic order (possibly $r = s$). □

1.2. Raney-type lemmas

Now using the above lemmas one easily proves, by reducing the sequence and just looking at it (the unicity is not needed) the following ones.

Lemma 4. *A sequence of reals with a positive sum has a splitting point.*

Lemma 5. *A sequence of integers with sum 1 has precisely one splitting point.*

Lemma 6. *A sequence of integers ≤ 1 with sum $s > 0$ has precisely s splitting points.*

Lemma 7. *A sequence of reals, all $\le r$ with sum $s > 0$, has at least $\lceil \frac{s}{r} \rceil$ splitting points.*

Remarks. A common graphical way to prove Lemma 5 is the "mountain and valley" method: draw the points $(k, \sum_1^k a_i)$ and find the rightmost among the lowest ones. Fig. 1 shows how to detect the splitting point 5 of the sequence −1, 2, 0, −2, 3, 0, −1. Note that the other lowest point reveals the weak splitting point 2.

It is common practice to call Lemma 5 Raney's Lemma, e.g. in [5], pages 345–346, where Lemma 6 is called a generalization ([5], page 348). We remark that in fact Lemma 6 is the one that is equivalent to Theorem 2.1 in Raney's paper [9]. The latter reads, a bit modified: a sequence of integers, all ≥ -1 with negative sum $-s$, admits precisely s cyclic shifts for which all partial sums except the final one are $> -s$. The equivalence is seen by changing all signs and reversing the sequence (each of which apart gives another equivalent lemma).

The well-known Cycle Lemma (on the p special breaking points in cyclic arrangements of k ones and $qk + p$ zeros) appears as a special case of Lemma 6 by replacing every 1 by −1 and every 0 by $\frac{1}{q}$. For extensions of the Cycle Lemma and related problems see [10] and its references.

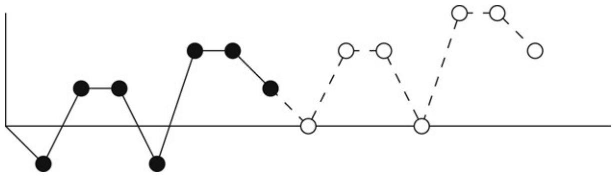

Fig. 1. Mountain and valley for −1, 2, 0, −2, 3, 0, −1.

Reducing the reversed sequence and using Lemma 1(d) one can prove Lemma 1 on page 364 in [4] (Ch. X11, Section 6), which reads as follows. A sequence with sum $S > 0$ has a cyclic shift for which all partial sums are $< S$; the number of such shifts equals the number of positions in such a shift where the partial sum is larger than the preceding partial sums.

Reducing a sequence (e.g. as in the proof of Lemma 3) provides an easy algorithm for finding the splitting points of a sequence with a positive sum (so if the process yields a reduced sequence of type 3). But see the second of the following examples.

Examples. Reducing $-2, -2, 4, -3, 3, -1, 0, 2, 1$ to $-4, 0, 1, 0, 2, 0, 0, 2, 1$ and further to $0, 0, 1, 0, 1, 0, 0, 0, 0$ shows that the splitting points are 3 and 5. The sequence $2, -1, -2, -2, 2$ with sum -1 has a splitting point 5 and can, by pushing, be transformed into $2, -3, 0, -2, 2$ with the same splitting point. But a push at 4 (the position preceding the splitting point, where the element -2 is (necessarily) smaller than the sum -1) gives $2, -3, -2, 0, 2$, without splitting points. However the final 2 would still be at a splitting point if we would eliminate the 0 created by the push. Unfortunately we would then lose track of the original positions of the splitting points.

In the next section we shall solve this problem by introducing "marked" zeros and see that, with a small modification, the algorithm can be used to find the splitting points and also the weak splitting points of arbitrary sequences.

2. Splitting points of arbitrary sequences: An algorithm

Suppose our sequence $A = a_1, a_2, \ldots, a_n$ has sum S, and a push is done resulting in $B = b_1, b_2, \ldots, b_n$. It is convenient and without loss of generality (use a cyclic shift) to suppose that the push is at n, so $b_n = 0$. As in the proof of Lemma 2 we find that the splitting points $\neq 1$ as well as the weak splitting points $\neq 1$ are the same in both sequences. However 1 is not a splitting point (a weak splitting point, respectively) of B if $S \leq 0$ (if $S < 0$, respectively), but may be one of A. It is one of A if and only if it is one of $C = b_1, b_2, \ldots, b_{n-1}$ and $S - a_n > 0$ ($S - a_n \geq 0$, respectively). Note that $S - a_n > 0$ is guaranteed if $S \geq 0$. Since j, with $j \neq 1$, is a (weak) splitting point of B if and only if it is one of C, we have:

Lemma 8. *Let $a_1, a_2, \ldots, a_n$, $n > 1$, be a sequence with sum S and let $a_n < 0$. Its splitting points $\neq 1$ and its weak splitting points $\neq 1$ are the same as those of $a_1, a_2, \ldots, a_{n-2}, a_{n-1} + a_n$. The first sequence has 1 as a splitting point (a weak splitting point, respectively) if and only if it is one of the second sequence and $S - a_n > 0$ ($S - a_n \geq 0$, respectively).*

Removal of the zero created by a push at i with $i < n$ would change the indices $> i$ of the elements in the sequence; to avoid this (it would be inconvenient, in particular in a computer program) we "mark" (the position of) the created zero and neglect it when examining the changed sequence.

To find the (weak) splitting points of a sequence one thus can transform it into a reduced sequence, keeping track of the marked zeros (notation $\underline{0}$) and moreover when pushing at i and $S - a_i \leq 0$ (< 0, respectively) excluding the first unmarked position after i (in cyclic order) as a splitting point (weak splitting point, respectively). By Lemma 1(a) S need only be calculated once. Note that if we proceed as in the proof of Lemma 3 after the first sweep all elements except possibly the first one are non-negative. Only if the first element is negative and $\neq S$ we must carry out the switch and start the second sweep. During the second sweep there always is at most one negative element and all elements to the right of it are marked zeros. When the negative element disappears the sequence is reduced (see Example(d) below). When it equals the sum of the sequence we can stop: all other elements are 0 or $\underline{0}$ and the only candidate for the unique (weak) splitting point is the first unmarked successor (in cyclic order) of the negative element (which could be the element itself, as in Example(a) below).

Note that one can push at a point directly after a marked zero, neglecting the mark ($\ldots, \underline{0}, -5, \ldots$ becomes $\ldots, -5, \underline{0}, \ldots$), the marked zero and the negative element then just change places. None of them is a weak splitting point. There is no need to unmark the new position of the -5: we are in the second sweep and there are positive elements to the left of the -5 (else we would have stopped before that push), so a new push at that position follows (see Example(e)).

Examples (*The Marked Zeros are Underlined*).

(a) $4, -3, 2, -1, -3$. First sweep: $-1, \underline{0}, \underline{0}, \underline{0}, \underline{0}$, splitting point 1.
(b) $0, -2, 0, 1, -1$. First sweep: $-2, \underline{0}, 0, 0, \underline{0}$; since $S - a_5 < 0$ index 1 is declared "no weak split"; since $S - a_2 = 0$ index 3 is declared "no split". We can stop now: weak splitting point 3. (The switch and a second sweep would lead to $\underline{0}, \underline{0}, -2, \underline{0}, \underline{0}$, with the same result.)
(c) $-2, 1, -2, 0, 1$. First step: $-2, -1, \underline{0}, 0, 1$; index 4 declared "no split". Next: $-3, \underline{0}, \underline{0}, 0, 1$; index 4 (the successor of index 2) declared "no weak split". Then $\underline{0}, \underline{0}, \underline{0}, 0, -2$. No (weak) splitting points, the (unique) candidate (index 4) is forbidden.
(d) $1, 1, -3, 0, 1$. First sweep: $-1, \underline{0}, \underline{0}, 0, 1$. Switch: $\underline{0}, \underline{0}, \underline{0}, 0, 0$. Weak splitting points 4 and 5.
(e) $-5, 2, -1, 0$ first gives: $-5, 1, \underline{0}, 0$, no weak split at 4, then $\underline{0}, 1, \underline{0}, -5$, next $\underline{0}, 1, \underline{-5}, \underline{0}$, and finally $\underline{0}, -4, \underline{0}, \underline{0}$. Splitting point 2.

Note that a sequence with sum 0 is reduced by the algorithm to a sequence of zeros. The weak splitting points are the indices of the non-underlined zeros. If and only if there is but one such a zero there is a splitting point. The weak splitting points partition the sequence into subsequences having sum 0 whereas their other non-empty partial sums are positive. One easily proves that these are the only subsequences with this property.

In the Appendix we give the algorithm in Maple program language.

3. Mountain and valley

Let $a_1, a_2, \ldots, a_n$ be a sequence with sum S (positive, negative or 0). Let $k > 1$. Now k is a splitting point if and only if

$$\begin{cases} \sum_{i=k}^{j} a_i > 0 & \text{for } k \le j \le n; \\ \sum_{i=k}^{n} a_i + \sum_{i=1}^{j} a_i > 0 & \text{for } 1 \le j \le k-2. \end{cases}$$

Rewrite these conditions as:

$$\begin{cases} \sum_{i=1}^{k-1} a_i < \sum_{i=1}^{j} a_i & \text{for } k \le j \le n; \\ \sum_{i=1}^{k-1} a_i < S + \sum_{i=1}^{j} a_i & \text{for } 1 \le j \le k-2. \end{cases}$$

For $k = 1$ the condition is $\sum_{i=1}^{j} a_i > 0$ for $1 \le j \le n-1$, rewritten as $S < S + \sum_{i=1}^{j} a_i$ for $1 \le j \le n-1$.

Now let, for $1 \le k \le n$, $P_k = \sum_{i=1}^{k} a_i$ and $P_{n+k} = S + P_k$. Then the conditions above precisely mean that k is a splitting point if and only if in the sequence $P_1, P_2, \ldots, P_{2n}$ the number P_{k-1} is smaller than the $n-1$ numbers following it (if $k = 1$ we must read P_0 as P_n).

For weak splitting points just change $>$ into $\ge$, $<$ into $\le$ and "smaller than" into "smaller than or equal to".

In fact this is nothing more than the "mountain and valley" method: a copy of the graph of the partial sums of the sequence is shifted such that its first point coincides with the final point.

This leads to an easy algorithm. Starting with $i = 1$ and ending with n we compare P_i with $P_{i+1}, P_{i+2}, \ldots,$ until we find an index j with $P_j \le P_i$ (for the weak case: $P_j < P_i$) or have reached P_{i+n-1} without finding such a number.

In the first case we take j as the next starting point. In the second case $i+1 \bmod n$ is a (weak) splitting point; if t is the greatest (in the weak case: the smallest) index in $\{i+1, i+2, \dots, i+n-1\}$ with $P_t = \min(P_{i+1}, P_{i+2}, \dots P_{i+n-1})$ we can take t (if still $\le n$) as the next starting point. Also we then know already that $P_t < P_j$ (in the weak case: $P_t \le P_j$) for $j = t+1, t+2, \dots, i+n-1$, but to use this we would also need to know the minimum of these P_j, and, for the next starting point, the minimum of the P_r coming after that minimum, etc.

The complexity of this algorithm seems to be $O(n^2)$, whereas that of the previous section was $O(n)$.

Example. The sequence 1, 2, 2, −1, −2, −1 has 1, 3, 5, 4, 2, 1, 2, 4, 6, 5, 3, 2 as its P-sequence. P_1 is not larger than the 5 elements following it, so 2 is a weak splitting point. P_6 is smaller than the 5 elements following it, so 1 is a splitting point.

Remark. Suppose we relax the definition of "splitting point" by excluding, apart from the empty partial sum, the last m partial sums (so 1 is called a splitting point of $a_1, a_2, \dots, a_n$ if $\sum_1^j a_i > 0$ for $1 \le j \le n-m$). As above one proves that k is a splitting point if and only if $P_{k-1} < P_j$ for $k \le j \le k+n-m-1$ (for $k=1$ read: $P_n < P_j$ for $n+1 \le j \le 2n-m$). For weak splitting points read $\le$ instead of $<$.

4. Clustering balls and Catalan numbers

4.1. Clustering balls

Suppose you stand on the edge of a pool, facing it, and draw balls from an urn containing n red balls and n blue ones (no replacement). Upon drawing a red (blue) ball you have to take one step forward (backward). The probability that you don't fall in is $\frac{1}{n+1}$ as is well known. A proof involves the calculation of the Catalan number C_n, the number of ± 1-sequences of length $2n$ with sum 0 and all non-empty partial sums ≥ 0. There are many proofs that $C_n = \frac{1}{n+1}\binom{2n}{n}$, as well as many combinatorial interpretations of C_n (e.g. see [11]). A common proof observes that C_n is also equal to the number of ± 1-sequences of length $2n+1$ with sum 1 and all non-empty partial sums > 0 (these have a 1 in front), and then counts using Lemma 5: $\frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}$ ([5], page 346).

Suppose now we "cluster" blue balls: a subset of k old blue balls is replaced by one new blue ball of "value" k, having the same probability of being drawn as the other balls, but permitting k steps backwards. This may be repeated for other subsets of old blue balls, not necessarily of the same cardinality k. A conjecture of W. Gielen (oral communication) stated that the probability of staying dry is unaltered by this (which is obvious if one clusters all blue balls into one). We prove a generalization of this conjecture to be true, but in fact that turns out to be equivalent to Raney's Theorem 2.2 in [9], which was proved by a counting argument using Lemma 6 (Theorem 2.1 in [9]).

4.2. A recursion formula

Suppose we have red balls $r_1, r_2, \dots, r_p$, each of value $+1$, and blue balls $b_1, b_2, \dots, b_q$ that have non-positive integer values $-v_1, -v_2, \dots, -v_q$, respectively, with $p, q \ge 0$ and $v := \sum_{i=1}^q v_i \le p$. The balls are well distinguished, so we have $(p+q)!$ sequences of balls, each having sum (of the values) $p-v$. Such a sequence is called *good* if all its partial sums are $\le p-v$.

Let $G(p; v_1, v_2, \dots, v_q)$ be the number of good sequences. Of course G is symmetric in $v_1, v_2 \dots, v_q$. Then $\frac{1}{(p+q)!}G(p; v_1, v_2, \dots, v_q)$ is the probability of staying dry when starting $p-v$ steps from the edge. We shall prove that it is $\frac{p+1-v}{p+1}$, thus proving Gielen's conjecture (the case $p=v$).

First suppose that at least one of the v_i is positive, v_q say. In a good sequence b_q is not the last ball, and we partition the set of good sequences: R_i ($i = 1, \dots, p$) consists of the sequences in which b_q is followed by r_i, B_j ($j = 1, \dots, q-1$) consists of those in which b_q is followed by b_j (none if $q=1$). Clearly all R_i have the same cardinality. By clustering b_q and r_p we get a bijection from R_p onto the set of good sequences consisting of $r_1, r_2, \dots, r_{p-1}$ and $b_1, b_2, \dots, b_{q-1}, b_q'$, where b_q' is a ball of value $-v_q+1$. So

$$|R_i| = G(p-1; v_1, v_2, \dots, v_{q-1}, v_q - 1) \quad \text{for } 1 \le i \le p.$$

Likewise we see by clustering b_q and b_j that

$$|B_j| = G(p; v_1, \ldots, \widehat{v_j}, \ldots, v_{q-1}, v_q + v_j) \quad \text{for } j = 1, \ldots, q-1.$$

($\widehat{v_j}$ means that v_j is omitted).

So we have, for $q > 0$ and $v_q > 0$:

$$G(p; v_1, v_2, \ldots, v_q) = pG(p-1; v_1, v_2, \ldots, v_{q-1}, v_q - 1) + \sum_{j=1}^{q-1} G(p; v_1, \ldots, \widehat{v_j}, \ldots, v_{q-1}, v_q + v_j). \quad (1)$$

Theorem 9. *$G(p; v_1, v_2, \ldots v_q) = \frac{p+1-v}{p+1}(p+q)!$ with $v = \sum_{i=1}^{q} v_i \leq p$.*

Proof. Induction on $p+q$, using (1), the symmetry in $v_1, \ldots, v_q$, and the trivial cases $G(p; \overbrace{0, \ldots, 0}^{q}) = (p+q)!$. □

4.3. Related well-known results

By a (lattice-)path we mean a sequence of lattice points in which a point (a, b) is followed by either $(a+1, b)$ or $(a, b+1)$. Such a path is called *p-good* if all its points lie below the line $y = (p-1)x$. The following corollary is Theorem 2.3 in [6].

Corollary 10. *Let p and q be integers with $p > q$ and $p > 1$. The number of p-good paths from $(1, q-1)$ to $(k, (p-1)k-1)$ is $\frac{p-q}{pk-q}\binom{pk-q}{k-1}$.*

Proof. By writing $-(p-1)$ for each horizontal step and $+1$ for each vertical step we see that every path from $(1, q-1)$ to $(k, (p-1)k-1)$ can be encoded by a sequence of $(p-1)k-q$ terms $+1$ and $k-1$ terms $-(p-1)$, so with sum $p-1-q$. That the path must stay below $y = (p-1)x$ means that an initial part of it cannot consist of s horizontal steps and more than $(s+1)(p-1)-q$ vertical steps, which means that in its encoding sequence all partial sums are $\leq p-1-q$. By Theorem 9 the number of such sequences is

$$\begin{aligned}
&\frac{1}{(k-1)!((p-1)k-q)!} G((p-1)k-q; \overbrace{p-1, \ldots, p-1}^{k-1}) \\
&= \frac{1}{(k-1)!((p-1)k-q)!} \cdot \frac{p-q}{(p-1)k-q+1}(pk-q-1)! \\
&= \frac{p-q}{(p-1)k-q+1}\binom{pk-q-1}{k-1} = \frac{p-q}{pk-q}\binom{pk-q}{k-1}. \quad \square
\end{aligned}$$

The following corollary is equivalent to the Cycle Lemma, see the Second Proof in [3]. It is used there to count rooted ordered $(m+1)$-ary forests.

Corollary 11. *The number of sequences with n terms $-m$, all other terms $+1$, sum $s > 0$ and (thus) length $L = mn + n + s$, of which all non-empty partial sums are positive, is $\frac{s}{L}\binom{L}{n}$.*

Proof. Such a sequence has a leading $+1$. Removing it and reading the sequence backwards we see that we can as well count the sequences with n terms $-m$ and $mn + s - 1$ terms $+1$ with all partial sums $\leq s-1$. Their number is

$$\begin{aligned}
&\frac{1}{(mn+s-1)!n!} G(mn+s-1; \overbrace{m, \ldots, m}^{n}) \\
&= \frac{1}{(mn+s-1)!n!} \cdot \frac{s}{mn+s}(mn+s-1+n)! = \frac{s}{L-n}\binom{L-1}{n} = \frac{s}{L}\binom{L}{n}. \quad \square
\end{aligned}$$

The case $s = 1$ is that of the generalized Catalan numbers $C_n^{(m+1)} = \frac{1}{mn+n+1}\binom{mn+n+1}{n} = \frac{1}{mn+1}\binom{mn+n}{n} = \frac{1}{n}\binom{mn+n}{n-1}$. (Note that $C_n = C_n^{(2)}$.) It corresponds to the instance of the "pool problem" in 4.2 in which $p = v$ and the blue balls are glued in equal clusters.

In [5], page 347, the numbers $C_n^{(m)}$ are called Fuss–Catalan numbers (after N. von Fuss, 1798!, no factorial) and Lemma 5 is used to prove the formula. Also Lemma 6 is used to find the formula in Corollary 11 (page 348). In the next section we shall show that the same counting method can be used to deal with sequences in which the negative terms need not be all equal, but that Theorem 9 could also do the job by a proof similar to that of Corollary 11.

We recall that $C_n^{(m+1)}$ is also the number of ways to divide a convex $(mn+2)$-polygon into $n(m+2)$-polygons by non-intersecting diagonals, probably the oldest application (Euler, Segner, Pfaff, von Fuss, see [2] and its references; also, the other ways to divide a polygon into polygons are discussed such that $C_n^{(m+1)}$ is the special case $D_{0,m(n-1)}^{(m+2)}$). $C_n^{(3)}$ is the number of trees of non-crossing diagonals of a $(n+1)$-polygon, see [8].

We get another special case by taking $s = m+1$ in Corollary 11. Removing the leading $m+1$ terms $+1$, reversing the sequences and replacing the $-m$'s by $+1$'s and the $+1$'s by 0's we establish a one-to-one correspondence with the 0,1-sequences of length $(m+1)n$ of which every initial part of length $j(m+1)$, $j = 1, 2, \ldots, n$, contains $\geq j$ terms 1. In [7] these sequences (forming a set $B(n, m+1)$) are shown to provide an encoding of the plane planted trees with $(m+1)(n+1)+2$ vertices and degrees 1 and $m+2$.

(Particular cases of) the numbers in Corollary 11 occur frequently, e.g. in the counting of parenthesis orders, ballot sequences, trees and lattice paths. See [1–3,6], the website [12] and their references/bibliographies.

4.4. Remarks

That balls of equal value are kept distinguished is needed in the proof of Theorem 9. The use of "fake" balls (value 0) is also essential, otherwise we would need Catalan as base of induction instead of proving it.

By reversing the order of the sequences we see that the number of good sequences equals that of the sequences with all partial sums ≥ 0. This means that as far as the probability of staying dry is concerned starting $p - v$ steps from the pool is the same as starting at the edge with your back turned to the pool. Reversing the order and changing all signs shows that in the pool problem of 4.1 (where $p = v$) we may as well cluster the red balls instead of the blue ones, but not both (confer balls 1, 1, -1, -1 with balls 2, -2). If $p > v$ clustering the red balls leads to different probabilities (try balls 1, 1, 1, -1, -1).

Finally note that a direct argument that clustering is allowed would reduce the proof of Theorem 9 to that for the case $q = 1$.

One can reformulate Raney's Theorem 2.2 in [9] (on "lists of words") as a theorem about sequences of balls in which balls of the same value are not distinguished, as follows (replace his values i by $i - 1$).

When there are m balls, a_i of which have value $i - 1$ for $i = 0, 1, 2\ldots$, with total value $-n < 0$, then the number of sequences with all partial sums except the final one $> -n$ is $\frac{n}{m} \cdot \frac{m!}{a_0! \cdot a_1! \cdot a_2! \cdot \cdots}$.

Of course this is the same as saying that with well-distinguished balls there are $n(m-1)!$ such sequences. Since the last ball is one of the a_0 balls of value -1, we see that by omitting it and changing all signs we get a_0 sets of good sequences (as defined in 4.1), with sum $n - 1 \geq 0$ and all partial sums $\leq n - 1$. These sets have cardinality $G(p; v_1, \ldots, v_q)$, where $p = a_0 - 1$, $q = m - a_0$, and a_i of the v_j are $i - 1$, $i = 1, 2, \ldots$. So that cardinality must be $\frac{n(m-1)!}{a_0}$. This is precisely what is stated in Theorem 9, since $n = a_0 - \sum v_i$.

5. Generalizing generalized Catalan numbers

We will count the number of "good" sequences with sum > 0 in which, apart from $+1$'s, different non-positive numbers may occur. We give two proofs. In the first one we find the number as a corollary of Theorem 9. The method in the second one is that used e.g. in [5,11] to prove special instances of Corollary 11 (and in fact has been used already by Raney [9]). In the second proof "balls" of equal value are not to be distinguished.

We consider the rearrangements of a sequence $x_1, x_2, \ldots, x_j, y_1, y_2, \ldots, y_k$ with $x_i = 1$ for $i = 1, 2, \ldots, j$, the y_i integers with $y_i \leq 0$ for $i = 1, 2, \ldots, k$ and $s = \sum_{i=1}^{j} x_i + \sum_{i=1}^{k} y_i > 0$. Suppose among the y_i there are l different numbers and let $f_1, f_2, \ldots, f_l$ be the frequencies with which they occur.

Theorem 12. *Let $z_1, z_2, \ldots, z_l$ be integers ≤ 0 and let $f_1, f_2, \ldots, f_l$ be non-negative integers. Let $k = \sum_{i=1}^{l} f_i$. The number of sequences consisting of j terms 1 and f_i terms z_i for $i \in \{1, 2, \ldots, l\}$, with positive sum s, and of which all non-empty partial sums are positive, is $\frac{s}{j+k}\binom{j+k}{j, f_1, f_2, \ldots, f_l}$.*

Proof 1. By omitting the $+1$ with which a good sequence must start and reversing the sequence we see that their number equals that of the sequences with $j-1$ terms $+1$ and f_i terms z_i $(i = 1, \ldots, l)$ of which all partial sums are $\leq s-1$. Applying Theorem 9 with $p = j-1$, $q = k$ and $v = -\sum f_i z_i$, realizing that now balls of equal value cannot be distinguished, we find $\frac{1}{(j-1)! f_1! \cdots f_l!} \cdot \frac{j + \sum f_i z_i}{j} \cdot (j-1+k)! = \frac{s}{j+k}\binom{j+k}{j, f_1, \cdots, f_l}$. $\square$

Proof 2. The number of rearrangements is $\binom{j+k}{j, f_1, f_2, \ldots, f_l}$, a multinomial coefficient. With each rearrangement we make the "block" (a multiset) containing all $j+k$ cyclic shifts of it. A rearrangement with period t then occurs in t of these blocks, and in every such block it occurs $\frac{j+k}{t}$ times. So in all blocks together every rearrangement occurs $j+k$ times. Now by Lemma 6 in a block we meet s times a sequence with positive non-empty partial sums. So the number of such sequences is $\frac{s}{j+k}\binom{j+k}{j, f_1, f_2, \ldots, f_l}$ (and the corresponding probability is $\frac{s}{j+k}$). $\square$

It is remarkable that the result does not depend on the particular numbers z_i, but only on their sum $\sum_{i=1}^{l} f_i z_i = s - j$. Even the frequencies f_i disappear if we take the sequence as a sequence of well-distinguished balls carrying the numbers; multiply the number in Theorem 12 by $j! \cdot f_1! \cdot f_2! \cdots f_l!$ to get:

Corollary 13. *The number of sequences of L well-distinguished balls containing j balls of value 1 and $L-j$ balls of non-positive integer values, with sum $s > 0$ and having all non-empty partial sums > 0, is $s(L-1)!$*

The sequences in the corollary start with one of the j balls of value 1. Take those that start with a fixed one of these balls, omit that ball and reverse the sequence to get (replacing $L-1$, $j-1$, and $s-1$ by L, j and s, respectively:

Corollary 14. *The number of sequences of L well-distinguished balls containing j balls of value +1 and $L-j$ balls of non-positive integer values, with sum $s \geq 0$ and having all partial sums $\leq s$, is $\frac{s+1}{j+1}L!$.*

This is Theorem 9 again.

Appendix. The Maple program "Splitpoints"

First the variables. `S` is the sum, `split` becomes the list of splitting points, `weaksplit` that of only weak splitting points. The array `succ`(-essor) holds for every position the next unmarked one and `foll`(-owing) moves through the unmarked positions. The forbidden positions are administrated in `nospl` and `noweakspl` and `last` is the position where we stopped pushing in the switch or in the second sweep. For the case `S >=0` these are not needed and a much smaller program would suffice.

```
splitpoints:= proc (T) # the sequence as a list T
   local Q, n, i, S, succ, mark, foll, last,
   split, weaksplit, nospl, noweakspl;
   Q := T; S:= 0; n:= nops(Q); mark:= array(1 .. n);
   succ:= array(1 ..n); split:= []; weaksplit:= [];
   for i to n do # initialize
      S:= S+Q[i]; succ[i]:= i+1; mark[i]:= false;
      nospl[i]:= false; noweakspl[i]:= false
   end do;
   succ[n]:= 1; # correction
   for i from n by -1 to 2 do # first sweep
      if Q[i]<0 then succ[i-1]:= succ[i]; mark[i]:= true;
         if S<0 then
           if S-Q[i]<0 then noweakspl[succ[i]]:= true
           elif S-Q[i]=0 then nospl[succ[i]]:= true
```

```
          end if
        end if;
        Q[i-1]:= Q[i-1]+Q[i]; Q[i]:= 0
      end if
   end do; # ready if Q[1]>=0
   if Q[1]<0 then
      if Q[1]=S then last:= 1 # ready
      else succ[n]:= succ[1]; mark[1]:= true; last:= n; # the switch
        if S<0 then
          if S-Q[1]<0 then noweakspl[succ[1]]:= true
          elif S-Q[1]=0 then nospl[succ[1]]:= true
          end if
        end if;
        Q[n]:= Q[n]+Q[1]; Q[1]:= 0; # now the second sweep
        for i from n by -1 to 2 while Q[i]<0 and Q[i]< S do
          succ[i-1]:= succ[i]; mark[i]:= true; last:= i-1;
          if S<0 then
            if S-Q[i]<0 then noweakspl[succ[i]]:= true
            elif S-Q[i]=0 then nospl[succ[i]]:= true
            end if
          end if;
          Q[i-1]:= Q[i-1]+Q[i]; Q[i]:= 0
        end do
      end if
   end if; # now determine the first unmarked position
   foll:= 1; for i to n while mark[i]=true do foll:= foll+1 end do;
   # make the list of (weak) splitting points
   if succ[foll]=foll then # there is only one unmarked position
      if nospl[foll]=false then split:= [op(split),foll]
      elif noweakspl[foll]=false then weaksplit:= [op(weaksplit),foll]
      end if
   elif S<0 then # Q[last] is the only non-zero element
      if noweakspl[succ[last]]=false then weaksplit:= [op(weaksplit),succ[last]]
      end if
   else # all elements are >=0
      while foll<succ[foll] do
        if 0<Q[foll] then
          split:= [op(split), foll] else weaksplit:= [op(weaksplit), foll]
        end if;
        foll:= succ[foll]
      end do; # now the last unmarked element
      if 0<Q[foll] then split:= [op(split), foll]
      else weaksplit:= [op(weaksplit), foll]
      end if
   end if;
   print(`SPLITPOINTS`);print(split); print(`ONLY WEAK SPLITPOINTS`);print(weaksplit)
end proc;
```

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